

AP Calculus BC

Chapter 12 Test Review – Differential Equations (All Section from Anton)

Section 7.7 – First Order Differential Equations

1. Separable
2. Linear: $y' + p(x)y = q(x)$
 - a. Let $\mu = e^{\int p(x)dx}$
 - b. Multiply both sides by μ .
 - c. Write in form $\frac{d}{dx} \mu y = \mu q(x)$
 - d. Integrate both sides, solve for y .

Section 14.1 – Second Order Differential Equations (Constant Coefficients)

1. Homogeneous: $y'' + py' + qy = 0$
2. Auxiliary/characteristic equation: $m^2 + pm + q = 0$
 - a. Two distinct real roots: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
 - b. One double root: $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$
 - c. Complex roots ($a + bi$): $y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

Section 14.2 – Second Order Differential Equations (Constant Coefficients)

1. Nonhomogeneous: $y'' + py' + qy = r(x)$
2. Complementary equation: $y'' + py' + qy = 0$; Complementary solution: $y_c(x)$
3. General solution = Complementary solution + Particular solution: $y = y_c(x) + y_p(x)$
4. Method of **undetermined coefficients**: choose y_p consistent with $r(x)$
 - a. If $r(x) = ke^{mx}$, choose $y_p(x) = Ae^{mx}$
 - b. If $r(x) = a_0 + a_1 x + \dots + a_n x^n$, choose $y_p(x) = A_0 + A_1 x + \dots + A_n x^n$
 - c. If $r(x) = c_1 \cos bx + c_2 \sin bx$, choose $y_p(x) = A_1 \cos bx + A_2 \sin bx$
 - d. You may have to multiply by x to avoid linear dependence.

Section 14.3 – Second Order Differential Equations (Constant Coefficients)

1. Nonhomogeneous: $y'' + py' + qy = r(x)$
2. **Variation of parameters**: general solution is $y = y_c(x) + y_p(x)$
 - a. Complementary solution: $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$
 - b. Particular solution: $y_p = u y_1 + v y_2$

c. $u' = \begin{vmatrix} 0 & y_2 \\ r(x) & y'_2 \end{vmatrix}; v' = \begin{vmatrix} y_1 & 0 \\ y'_1 & r(x) \end{vmatrix}$

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y'_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}}; v' = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & r(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}}$$

Practice: 7.7: #10, 22; 14.1: #12, 18; 14.2: #4, 20; 14.3: #4, 18

CHAPTER 12 TEST REVIEW

SECTION 7.7

$$(10) \quad 2\frac{dy}{dx} + 4y = 1 \quad \mu = e^{\int 2dx} = e^{2x}$$

$$\frac{dy}{dx} + 2y = \frac{1}{2}$$

$$\frac{d}{dx}(e^{2x} \cdot y) = \frac{1}{2}e^{2x} \Rightarrow e^{2x} \cdot y = \frac{1}{4}e^{2x} + C \Rightarrow y = \frac{1}{4} + Ce^{-2x}$$

$$(22) \quad \frac{dy}{dx} = 2y + 3 \quad (1,-1)$$

$$\frac{1}{2y+3} dy = dx$$

$$\frac{1}{2} \ln|2y+3| = x + C$$

$$\ln|2y+3| = 2x + C$$

$$2y+3 = Ce^{2x}$$

$$2(-1) + 3 = Ce^2$$

$$1 = Ce^2$$

$$C = e^{-2}$$

$$2y+3 = e^{-2} \cdot e^{2x} = e^{2x-2}$$

$$y = \frac{1}{2}(e^{2x-2} - 3)$$

SECTION 14.1

$$(12) \quad y'' - 10y' + 25y = 0$$

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0$$

$$y_c = c_1 e^{5t} + c_2 t e^{5t}$$

$$(18) \quad y'' - 6y' - 7y = 0 \quad y(0) = 5 \quad y'(0) = 3$$

$$m^2 - 6m - 7 = 0$$

$$(m-7)(m+1) = 0$$

$$y_c = c_1 e^{7x} + c_2 e^{-x} \Rightarrow 5 = c_1 + c_2$$

$$y'_c = 7c_1 e^{7x} - c_2 e^{-x} \Rightarrow 3 = 7c_1 - c_2$$

$$\therefore y_c = e^{7x} + 4e^{-x}$$

$$\begin{array}{l} 8 = 8c_1 \\ c_1 = 1 \end{array} \quad \begin{array}{l} c_2 = 4 \end{array}$$

SECTION 14.2

$$\textcircled{4} \quad y'' + 7y' - 8y = 7e^x$$

$$m^2 + 7m - 8 = 0$$

$$(m+8)(m-1) = 0$$

$$y_c = c_1 e^{-8x} + c_2 e^x$$

$$y_p = Ax e^x$$

$$y_p' = Axe^x + Ae^x$$

$$y_p'' = Axe^x + Ae^x + Ae^x = Axe^x + 2Ae^x$$

~~$$Axe^x + 2Ae^x + 7Axe^x + 7Ae^x - 8Axe^x = 7e^x$$~~

$$9Ae^x = 7e^x$$

$$9A = 7$$

$$A = 7/9$$

$$y_p = \frac{7}{9}xe^x$$

$$\therefore y_g = c_1 e^{-8x} + c_2 e^x + \frac{7}{9}xe^x$$

$$\textcircled{20} \quad y'' + 4y' + 4y = 3x + 3$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0$$

$$y_c = c_1 e^{-2x} + c_2 xe^{-2x}$$

$$y_p = A_0 + A_1 x$$

$$y_p' = A_1$$

$$y_p'' = 0$$

$$0 + 4A_1 + 4(A_0 + A_1 x) = 3x + 3$$

$$\underline{4A_1} + \underline{4A_0} + \underline{4A_1 x} = \underline{3x + 3}$$

$$A_1 = 3/4$$

$$4A_1 + 4A_0 = 3$$

$$4(3/4) + 4A_0 = 3$$

$$4A_0 = 0$$

$$A_0 = 0$$

$$y_p = 0 + \frac{3}{4}x$$

$$\therefore y_g = c_1 e^{-2x} + c_2 xe^{-2x} + \frac{3}{4}x$$

SECTION 14.3

④ $y'' + 5y' + 6y = e^{-x}$

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

$$u' = \frac{\begin{vmatrix} 0 & e^{-3x} \\ e^{-x} & -3e^{-3x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}} = \frac{-e^{-4x}}{-3e^{-5x} + 2e^{-5x}} = \frac{-e^{-4x}}{-e^{-5x}} = e^x \Rightarrow u = e^x$$

$$\begin{cases} u'(e^{-2x}) + v'(e^{-3x}) = 0 \\ u'(-2e^{-2x}) + v'(-3e^{-3x}) = e^{-x} \end{cases} \quad v' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-x} \end{vmatrix}}{\begin{vmatrix} -e^{-5x} & 0 \\ -e^{-5x} & -e^{-5x} \end{vmatrix}} = \frac{e^{-3x}}{-e^{-5x}} = -e^{2x} \Rightarrow u = -\frac{1}{2} e^{2x}$$

$$y_p = e^x(e^{-2x}) + -\frac{1}{2} e^{2x}(e^{-3x}) \\ = e^{-x} - \frac{1}{2} e^{-x}$$

$$y_p = \frac{1}{2} e^{-x}$$

$$\therefore y_g = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{2} e^{-x}$$

⑧ $y'' - y = x^2 e^x$

$$m^2 - 1 = 0$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$u' = \frac{\begin{vmatrix} 0 & e^{-x} \\ x^2 e^x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-x^2}{-e^0 - e^0} = \frac{-x^2}{-2} = \frac{1}{2} x^2 \Rightarrow u = \frac{1}{6} x^3$$

$$u'e^x + v'e^{-x} = 0$$

$$u'e^x + v'(-e^{-x}) = x^2 e^x$$

$$v' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & x^2 e^x \end{vmatrix}}{-2} = \frac{x^2 e^{2x}}{-2} = -\frac{1}{2} x^2 e^{2x}$$

$$v = \int -\frac{1}{2} x^2 e^{2x} dx = -\frac{1}{4} x^2 e^{2x} + \frac{1}{4} x e^{2x} - \frac{1}{8} e^{2x}$$

$$+ \frac{u}{e^{2x}} \frac{dv}{e^{2x}}$$

$$y_p = \frac{1}{6} x^3 e^x + \left(-\frac{1}{4} x^2 e^{2x} + \frac{1}{4} x e^{2x} - \frac{1}{8} e^{2x} \right) \cdot e^{-x}$$

$$- -x \rightarrow \frac{1}{2} e^{2x}$$

$$+ -1 \rightarrow \frac{1}{4} e^{2x}$$

$$- 0 \rightarrow \frac{1}{8} e^{2x}$$

$$y_g = y_c + y_p$$